mathematical methods - week 4

Complex differentiation

Georgia Tech PHYS-6124

Homework HW #4 due Thursday, September 17, 2020

== show all your work for maximum credit,
== put labels, title, legends on any graphs
== acknowledge study group member, if collective effort
== if you are LaTeXing, here is the exerWeek4.tex

Exercise 4.2 *Complex arithmetic* 10 (+3 bonus) points
Exercise 4.5 *Circles and lines with complex numbers* 3 points

Bonus points
Exercise 4.1 *Complex arithmetic – principles* 6 points

Total of 13 points = 100 % score. Extra points accumulate, can help you later if you miss a few problems.

edited September 13, 2020

Week 4 syllabus

Tuesday, September 8, 2020

This week's lectures are related to Arfken, Weber & Harris [2] Chapter 11 *Complex variable theory* (click here).

*Complex numbers : full **Tue** lecture* - Complex variables; History; algebraic and geometric insights; De Moivre's formula; roots of unity; functions of complex variables as mappings (the last 7 minutes were not in the live lecture)

AWH 11.1 *Complex variables and functions*

*Cauchy-Riemann : full **Thu** lecture* - differentiation of complex functions; Cauchy-Riemann conditions; holomorphic (analytic) functions; conformal mappings (includes a conformal mapping clip that was not in the live lecture)

AWH 11.2 *Cauchy-Riemann conditions*

Everything is allowed in love and war (how to do problem sets)

Optional reading

- Grigoriev notes pages 2.1 - 2.3 (clean and concise)
- I personally am a big fan of Stone and Goldbart [4] (click here); our lectures on complex numbers follow Paul Goldbart's lectures.

  SG 17.1 *Cauchy-Riemann equations*

  SG 17.1.2 *Conformal mapping*

  SG 17.3.3 *Blasius and Kutta-Joukowski theorems* (for the rocket scientists among us)

  SG 17.6.1 *The point at infinity* (Riemann sphere)

- Ahlfors [1] (click here)
- Needham [3] (click here)

Alex Kontorovich, Rutgers MAT 640:503 Complex Analysis. A wonderful lecturer, here he diverges into the story of Cardano and cubics. They are *cube*-ic for a reason. Did you know people learned to use $\sqrt{-1}$ before they understood that a number can be negative, like -1? Listen to his first lecture. Oh no! He just made me solve the cubic, something I had avoided my entire life. So far. You'll love it.

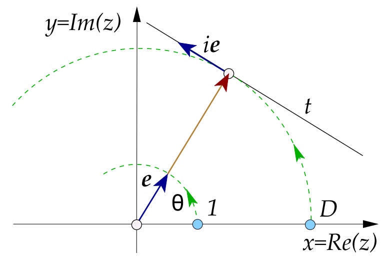

Figure 4.1: A unit vector $\mathbf{e}$ multiplied by a real number D traces out a circle of points in the complex plane. Multiplication by the imaginary unit i rotates a complex vector by 90^0, so $D\mathbf{e} + it\mathbf{e}$ is a tangent to this circle, a line parametrized by a real number t.

Question 4.1. Henriette Roux asks

Q You made us do exercise 4.5, but you did not cover this in class? I left it blank!

A Mhm. I told you that complex numbers can be understood as vectors in the complex plane, vectors that can be added and multiplied by scalars. I told you that the multiplication by the imaginary unit i rotates a complex vector by 90^0. I told you that in the polar representation, complex numbers define circle parametrized by their argument (phase). For example, a line is defined by its orientation $\mathbf{e}$, and its shortest distance to the origin is along the vector $D\mathbf{e}$, of length D, see figure 4.1.

The point of the exercise is that if you use your high school $\sin$'s and $\cos$'s, this simple formula (and the other that have to do with circles) is a mess.

References

[1] L. V. Ahlfors, *Complex Analysis*, 3rd ed. (Mc Graw Hill, 1979).

[2] G. B. Arfken, H. J. Weber, and F. E. Harris, *Mathematical Methods for Physicists: A Comprehensive Guide*, 7th ed. (Academic, New York, 2013).

[3] T. Needham, *Visual Complex Analysis* (Oxford Univ. Press, Oxford UK, 1997).

[4] M. Stone and P. Goldbart, *Mathematics for Physics: A Guided Tour for Graduate Students* (Cambridge Univ. Press, Cambridge UK, 2009).

Exercises

4.1. **Complex arithmetic – principles:** (Ahlfors [1], pp. 1-3, 6-8)

(a) (bonus) Show that $\frac{A+iB}{C+iD}$ is a complex number provided that $C^2 + D^2 \neq 0$. Show that an efficient way to compute a quotient is to multiply numerator and denominator by the conjugate of the denominator. Apply this scheme to compute the quotient $\frac{A+iB}{C+iD}$.

(b) (bonus) By considering the equation $(x + iy)^2 = (A + iB)$ for real x, y, A and B, compute the square root of $A + iB$ explicitly for the case $B \neq 0$. Repeat for the case $B = 0$. (To avoid confusion it is useful to adopt he convention that square roots of positive numbers have real signs.) Observe that the square root of any complex number exists and has two (in general complex) opposite values.

(c) (bonus) Show that $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$ and that $\overline{z_1\, z_2} = \bar{z}_1\, \bar{z}_2$. Hence show that $\overline{z_1/z_2} = \bar{z}_1/\bar{z}_2$. Note the more general result that for any rational operation R applied to the set of complex numbers $z_1, z_2, \ldots$ we have $\overline{R(z_1, z_2, \ldots)} = R(\bar{z}_1, \bar{z}_2, \ldots)$. Hence, show that if ζ solves $a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0 = 0$ then $\bar{\zeta}$ solves $\bar{a}_n z^n + \bar{a}_{n-1} z^{n-1} + \cdots + \bar{a}_0 = 0$.

(d) (bonus) Show that $|z_1\, z_2| = |z_1|\,|z_2|$. Note that this extends to arbitrary finite products $|z_1\, z_2\, \ldots| = |z_1|\,|z_2|\, \ldots$. Hence show that $|z_1/z_2| = |z_1|/|z_2|$. Show that $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 + 2\mathrm{Re}\, z_1\, \bar{z}_2$ and that $|z_1 - z_2|^2 = |z_1|^2 + |z_2|^2 - 2\mathrm{Re}\, z_1\, \bar{z}_2$.

4.2. **Complex arithmetic.** (Ahlfors [1], pp. 2-4, 6, 8, 9, 11)

(a) Find the values of

$$\big(1 + 2i\big)^3, \qquad \frac{5}{-3 + 4i}, \qquad \left(\frac{2 + i}{3 - 2i}\right),$$

$$\big(1 + i\big)^N + \big(1 - i\big)^N \quad \text{for} \quad N = 1, 2, 3, \ldots.$$

(b) If $z = x + iy$ (with x and y real), find the real and imaginary parts of

$$z^4, \qquad \frac{1}{z}, \qquad \frac{z - 1}{z + 1}, \qquad \frac{1}{z^2}.$$

(c) Show that, for all combinations of signs,

$$\left(\frac{-1 \pm i\sqrt{3}}{2}\right)^3 = 1, \qquad \left(\frac{\pm 1 \pm i\sqrt{3}}{2}\right)^6 = 1.$$

(d) By using their Cartesian representations, compute $\sqrt{i}$, $\sqrt{-i}$, $\sqrt{1+i}$ and $\sqrt{\frac{1 - i\sqrt{3}}{2}}$.

(e) By using the Cartesian representation, find the four values of $\sqrt[4]{-1}$.

(f) By using their Cartesian representations, compute $\sqrt[4]{i}$ and $\sqrt[4]{-i}$.

(g) Solve the following quadratic equation (with real A, B, C and D) for complex z:

$$z^2 + (A + iB)z + C + iD = 0.$$

(h) Show that the system of all matrices of the form

$$\begin{bmatrix} A & B \\ -B & A \end{bmatrix}$$

(with real A and B), when combined by matrix addition and matrix multiplication, is isomorphic to the field of complex numbers.

(i) Verify by calculation that the values of $z/\big(z^2 + 1\big)$ for $z = x + iy$ and $z = x - iy$ are conjugate.

(j) Find the absolute values of

$$-2i\big(3 + i\big)\big(2 + 4i\big)\big(1 + i\big), \qquad \frac{\big(3 + 4i\big)\big(-1 + 2i\big)}{\big(-1 - i\big)\big(3 - i\big)}.$$

(k) Prove that, for complex a and b, if either $|a| = 1$ or $|b| = 1$ then

$$\left|\frac{a-b}{1-\bar{a}b}\right| = 1.$$

What exception must be made if $|a| = |b| = 1$?

(l) Show that there are complex numbers z satisfying $|z-a| + |z+a| = 2|c|$ if and only if $|a| \le |c|$. If this condition is fulfilled, what are the smallest and largest values of $|z|$?

(m) Prove the complex form of Lagrange's identity, *viz.*, for complex $\{a_j, b_j\}$

$$\Big|\sum_{j=1}^{n} a_j\, b_j\Big|^2 = \sum_{j=1}^{n} |a_j|^2 \sum_{j=1}^{n} |b_j|^2 - \sum_{1\le j<k\le n} \big|a_j\, \bar{b}_k - a_k\, \bar{b}_j\big|^2 .$$

4.3. **Complex inequalities – principles:** (Ahlfors [1], pp. 9-11)

(a) (bonus) Show that $-|z| \le \operatorname{Re} z \le |z|$ and that $-|z| \le \operatorname{Im} z \le |z|$. When do the equalities $\operatorname{Re} z = |z|$ or $\operatorname{Im} z = |z|$ hold?

(b) (bonus) Derive the so-called triangle inequality $|z_1 + z_2| \le |z_1| + |z_2|$. Note that it extends to arbitrary sums: $|z_1 + z_2 + \cdots| \le |z_1| + |z_2| + \cdots$. Under what circumstances does the equality hold? Show that $|z_1 - z_2| \ge \big||z_1| - |z_2|\big|$.

(c) (bonus) Derive Cauchy's inequality, i.e., show that

$$\Big|\sum_{j=1}^{n} w_j\, z_j\Big|^2 \le \Big|\sum_{j=1}^{n} |w_j|^2 \Big|\sum_{j=1}^{n} |z_j|^2.$$

4.4. **Complex inequalities:** (Ahlfors [1], p. 11)

(a) (bonus) Prove that, for complex a and b such that $|a| < 1$ and $|b| < 1$, we have $|(a-b)/(1-\bar{a}b)| < 1$.

(b) (bonus) Let $\{a_j\}_{j=1}^{n}$ be a set of n complex variables and let $\{\lambda_j\}_{j=1}^{n}$ be a set of n real variables.
If $|a_j| < 1$, $\lambda_j \ge 0$ and $\sum_{j=1}^{n} \lambda_j = 1$, show that $\left|\sum_{j=1}^{n} \lambda_j\, a_j\right| < 1$.

4.5. **Circles and lines with complex numbers:** (Needham [3] p. 46)

(a) If c is a fixed complex number and R is a fixed real number, explain with a picture why $|z - c| = R$ is the equation of a circle. Given that z satisfies the equation $|z + 3 - 4i| = 2$, find the minimum and maximum values of $|z|$ and the corresponding positions of z.

(b) Consider the two straight lines in the complex plane that make an angle $(\pi/2) + \phi$ with the real axis and lie a distance D from the origin. Show that points z on the lines satisfy one or other of $\operatorname{Re}\big(\cos\phi - i\sin\phi\big)z = \pm D$.

(c) Consider the circle of points obeying $\big|z - (D+R)\big(\cos\phi + i\sin\phi\big)\big| = R$. Give the centre of this circle and its radius. Determine what happens to this circle in the $R \to \infty$ limit. (Note: In the *extended* complex plane the properties of circles and lines are unified. For this reason they are sometimes referred to as *circlines*.)

4.6. **Plane geometry with complex numbers:** (Ahlfors [1], p. 15)

(a) Prove that if the points a_1, a_2 and a_3 are the vertices of an equilateral triangle then $a_1\, a_1 + a_2\, a_2 + a_3\, a_3 = a_1\, a_2 + a_2\, a_3 + a_3\, a_1$.

(b) Suppose that a and b are two vertices of a square in the complex plane. Find the two other vertices in all possible cases.

(c) (bonus) Find the center and the radius of the circle that circumscribes the triangle having vertices a_1, a_2 and a_3. Express the result in symmetric form.

(d) (bonus) Find the symmetric points of the complex number z with respect to each of the lines that bisect the coordinate axes.

4.7. **More plane geometry with complex numbers:** (Needham [3] p. 16)
Consider the quadrilateral having *sides* given by the complex numbers $2a_1$, $2a_2$, $2a_3$ and $2a_4$, and construct the squares on these sides. Now consider the two line-segments joining the centres of squares on opposite sides of the quadrilateral. Show that these line-segments are perpendicular and of equal length.

4.8. **More plane geometry with complex numbers:** (Ahlfors [1], p. 9, 17)

(a) Find the conditions under which the equation $az + b\bar{z} + c = 0$ (with complex a, b and c) in one complex unknown z has exactly one solution, and compute that solution. When does the equation represent a line?

(b) (bonus) Write the equation of an ellipse, hyperbola and parabola in complex form.

(c) (bonus) Show, using complex numbers, that the diagonals of a parallelogram bisect each other.

(d) (bonus) Show, using complex numbers, that the diagonals of a rhombus are orthogonal.

(e) (bonus) Show that the midpoints of parallel chords to a circle lie on a diameter perpendicular to the chords.

(f) (bonus) Show that all circles that pass through a and $1/a$ intersect the circle $|z| = 1$ at right angles.

4.9. **Number theory with complex numbers:** (Needham [3] p. 45)
Here is a basic fact that has many uses in number theory: *If two integers can be expressed as the sum of two squares then so can their product*. Prove this result by considering $\big|(A + iB)(C + iD)\big|^2$ for integers A, B, C and D.

4.10. **Trigonometry with complex numbers:** (Ahlfors [1], pp. 16-17)

(a) Express $\cos 3\phi$, $\cos 4\phi$ and $\sin 5\phi$ in terms of $\cos\phi$ and $\sin\phi$.

(b) Simplify $1 + \cos\phi + \cos 2\phi + \cdots + \cos N\phi$ and $\sin\phi + \sin 2\phi + \sin 3\phi + \cdots + \sin N\phi$.

(c) Express the fifth and tenth roots of unity in algebraic form.

(d) (bonus) If ω is given by $\omega = \cos\big(2\pi/N\big) + i \sin\big(2\pi/N\big)$ (for $N = 0, 1, 2, \ldots$), show that, for any integer H that is not a multiple of N, $1 + \omega^H + \omega^{2H} + \cdots + \omega^{(N-1)H} = 0$. What is the value of $1 - \omega^H + \omega^{2H} - \cdots + (-1)^{N-1}\omega^{(N-1)H}$?