A review of the Hénon map and its physical interpretations

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The Hénon map is one of the simplest two-dimensional mappings exhibiting chaotic behavior. It has been extensively studied due to its low dimension and chaotic dynamics. Even though the Hénon map is introduced mathematically as a model problem and has no particular physical import of itself, links between certain harmonic oscillators and "Hénon-like" maps have been found. To get a better understanding of the Hénon map, we review the dynamic properties of the Hénon map including its fixed points, stability, periodic orbits, and so on, and a physical interpretation of it is discussed.

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I. INTRODUCTION

In 1963, Lorenz proposed a system of three coupled differential equations, which is later well known as the Lorenz flow. The Lorenz flow was first studied because it is of interest for weather prediction. However, further explorations of the Lorenz flow have revealed even more benefits. As an illustration of deterministic chaos, the Lorenz flow was widely explored and studied. Motivated by the Lorenz equations, Hénon introduced a simple two dimensional map in 1976 [1], which captured the stretching and folding dynamics of chaotic systems such as the Lorenz system.

The Hénon map is a minimal normal form for modeling flows near a saddle-node bifurcation, and it is a prototype of the stretching and folding dynamics that leads to deterministic chaos. Due to its simple form, the Hénon map has provide us a way to conduct more detailed exploration of the chaotic dynamics. Interesting enough, even though the Hénon map is introduced as a mathematical model, it still corresponds to the dynamics of some physical systems, one of which was given by Biham and Wenzel [2]. However, this interpretation is mostly exploited as a computational tool instead of an illustration of the physical dynamics. In contrast, Heagy demonstrated an interesting interpretation of the Hénon map [3]. This interpretation links the Hénon map to the period one return map of an impulsively driven harmonic oscillator, which provides a relatively good insight into the dynamics of the Hénon map.

In sect. II the standard form of the Hénon map is introduced and the details of its dynamic properties are discussed. A review of an physical interpretation of the Hénon map is shown in sect. III. In sect. IV the properties of the Hénon map are briefly summarized, and a discussion of its physical interpretations is given.

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II. THE HÉNON MAP

In 1969, Hénon showed in Ref [4] that essential properties of dynamical systems defined by differential equations can be retained by a carefully defined area-preserving mappings. Inspired by the same idea, Hénon proposed the famous two dimensional Hénon map as a reduced approach to study the dynamics of the Lorenz system. *The Hénon map* is given by the following equations:

$$\begin{aligned} x_{n+1} &= 1 - a x_n^2 + b y_n \\ y_{n+1} &= x_n \end{aligned} \tag{1}$$

This is a nonlinear two dimensional map, which can also be written as a two-step recurrence relation

$$x_{n+1} = 1 - a x_n^2 + b x_{n-1} \tag{2}$$

A. Fixed points and Hénon attractor

An *attractor* refers to a subset of a connected state space $\mathcal{M}_0$, where the flow is globally contracting onto, as $\mathcal{M}_0$ mapping into itself under forward evolutions. An attractor can be a fixed point, a periodic orbit, aperiodic, or a combination of the above. The most interesting case is the aperiodic recurrent attractor, which is also referred to as a *strange attractor*.

For the Hénon map, we have two fixed points. Taking $(x_{n+1}, y_{n+1}) = (x_n, y_n) = (x_0, x_0)$ in (1), we have

$$x_0 = \frac{-(1-b) \pm \sqrt{(1-b)^2 + 4a}}{2a} \tag{3}$$

which can be either attracting or saddle points depending on the choice of parameters (a, b) [5].

In Ref [1], Hénon had also claimed that for $(a, b) = (1.4, 0.3)$ the Hénon map converges to a strange attractor. A visualization of the Hénon attractor can be done

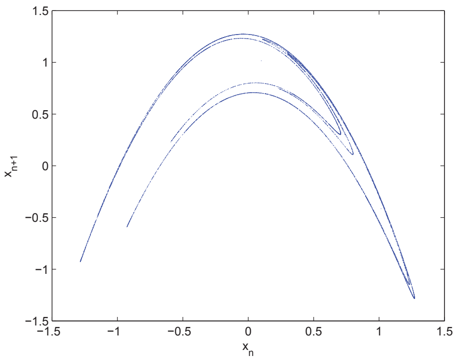
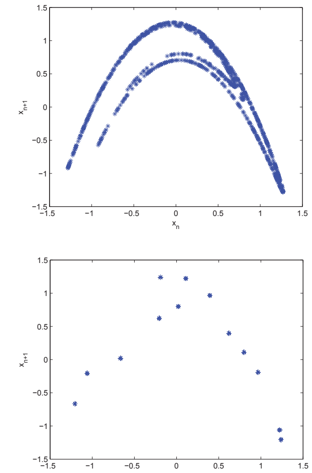


FIG. 1: A visualization of the Hénon attractor.

by numerical iterations. By picking an arbitrary initial point and iterating (1) on a computer and plotting the results on the (x_n, x_{n+1}) plane, we can get a sketch of the dynamics of the Hénon map. An iteration of 10,000 with initial point (0.1, 0.1) is plotted in Fig. 1. As we have mentioned, the Hénon map is one of the simplest maps capturing the stretching and folding dynamics of chaotic systems. The parameter a controls the amount of stretching and the parameter b controls the thickness of folding. In Fig. 1, b is relatively large and the attractor is rather thick, which gives a clearly visible transverse fractal structure.

It is worth mentioning that even though the Hénon attractor at $(a, b) = (1.4, 0.3)$ is shown to be a strange attractor for all practical purposes, the existence of the strange attractor has never been proven and the Hénon attractor could be a result of some long attracting stable cycles. Actually, it is possible to find stable attractors arbitrarily close by in the parameter space. An example is the 13-cycle attractor at $(a, b) = (1.39945219, 0.3)$. A comparison of the "Hénon attractor and the 13-cycle attractor is shown in Fig. 2.



(a)

(b)

FIG. 2: Plot of the Hénon attractor and the 13-cycle attractor. Figure obtained by iterating the Hénon map with specified values of (a, b), and removing and re-plotting points every 1,000 iterations. (a) The Hénon attractor at $(a, b) = (1.4, 0.3)$, and (b) the 13-cycle attractor at $(a, b) = (1.39945219, 0.3)$.

B. Jacobian matrix and stability

The Jacobian matrix is an important tool to explore the properties of dynamic systems. Evaluating it for the Hénon map we get the Jacobian matrix for the nth iterate is

$$J^n(x_0) = \prod_{m=n}^{1} \begin{pmatrix} -2ax_m & b \\ 1 & 0 \end{pmatrix} . \tag{4}$$

Calculating the determinant of the Hénon one time-step Jacobian matrix gives us a good insight into the dynamics of the map. From (4) we get

$$\det J = \Lambda_1 \Lambda_2 = -b \, , \tag{5}$$

which is a constant. This gives a constant area contraction rate for the Hénon map. The constant Jacobian is not an accident. Hénon had demonstrated this interesting property when he first introduced the Hénon map in Ref [1]. He claimed that by dividing the map into three steps: a folding, a contraction along the x axis, and a flipping about $x = y$, each step has a simple and unique geometrical interpretation, namely, the folding preserves areas, the flipping preserves the area but reverses the sign, and the contraction contracts the area by a constant factor b. Altogether, we have a area contraction rate of $-b$. This kind of map is known as an *entire Cremona transformation*, which is a one-by-one mapping of the plane into itself.

The Floquet matrix of the Hénon map can also be evaluated from the Jacobian matrix. By picking a periodic point as a starting point, and multiplying individual periodic point Jacobian matrices around a prime cycle, we get the Floquet matrix as

$$M_p(x_0) = \prod_{k=n_p}^{1} \begin{pmatrix} -2ax_k & b \\ 1 & 0 \end{pmatrix} , \qquad x_k \in \mathcal{M}_p \, . \tag{6}$$

For a specific Hénon map with (a, b) given, the Floquet multipliers and Floquet exponents can therefore be calculated, and the stability of periodic orbits can be evaluated accordingly (based on $|\Lambda_j|$ and the sign of the real part of the Floquet exponents $\mu^{(i)}$).

C. Hénon repeller and horseshoe

For the study of long-term dynamics, only the nonwandering set Ω of a dynamics system is of our interest, where the trajectories reenter the neighborhood infinitely often. A non-wandering set is the union of all kinds of separately invariant sets including attractors and repellers. In sect. II A we have already introduced the attractor, which is a subset of a connected state space attracting the flow globally. Conversely, for a non-wandering set Ω enclosed by a connected state space volume $\mathcal{M}_0$, if all points within $\mathcal{M}_0$ but not Ω will eventually exit $\mathcal{M}_0$, the non-wandering set Ω is called a *repeller*.

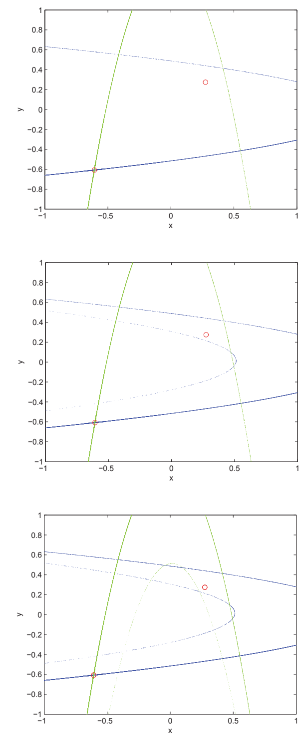
For the Hénon map with parameter $b \neq 0$, interesting repellers can be found. The Hénon map takes a rectangular area and returns it bent as a *Smale horseshoe*. The Hénon repeller with parameter $b = -1$ and a large parameter a is especially instructive. According to (5), for $b = -1$, the contraction rate is 1, which means the map is area preserving. At the same time, the map is strongly stretching due to a large parameter a. Therefore, we get a strongly stretching but yet area preserving map, where the folded horseshoe can be clearly observed and the stable manifold $\mathcal{W}^s$ and unstable manifold $\mathcal{W}^u$ of the can be studied visually. For parameter $(a, b) = (6, -1)$, different iterates of the map are calculated numerically and plotted on the (x, y) plane in Fig. 3.

Starting with a set of points in a small square around the fix point x_0, iterating forward using (1) will stretch and fold the initial set of points and trace out the unstable manifold $\mathcal{W}^u$, as indicated by the blue line in Fig. 3(a). On the other hand, the backward iteration in time is given by

$$\begin{aligned} x_{n-1} &= y_n \\ y_{n+1} &= -b^{-1}(1 - a y_n^2 - b x_n) \end{aligned} \tag{7}$$

Iterated backward in time, the initial set will outline the stable manifold $\mathcal{W}^s$, as indicated by the green line in Fig. 3(a). The intersection of $\mathcal{W}^s$ and $\mathcal{W}^u$ gives an invariant and optimal initial region $\mathcal{M}$ where the non-wandering set is enclosed. We say $\mathcal{M}$ is invariant and optimal because any point outside $\mathcal{W}^s$ border escapes to infinity forward in time and any point outside $\mathcal{W}^u$ border comes from infinity backward in time.

As we iterate one more step forward in time, $\mathcal{M}$ will be stretched and folded to form a Smale horseshoe, and the intersection is split into two future strips as shown in Fig. 3(b). Label the strips symbolically we have $\mathcal{M}_{0.}$ and $\mathcal{M}_{1.}$ respectively. Similarly, iterate one more step backward in time, the intersection becomes four regions as shown in Fig. 3(c), which can be labeled as $\mathcal{M}_{0.0}$, $\mathcal{M}_{0.1}$, $\mathcal{M}_{1.0}$ and $\mathcal{M}_{1.1}$. Iterate one more step forward, we will get 4 future strips labeled by $\mathcal{M}_{00.}$, $\mathcal{M}_{01.}$, $\mathcal{M}_{10.}$, $\mathcal{M}_{11.}$, and iterate one more step backward, we will get 4 past strips labeled by $\mathcal{M}_{.00}$, $\mathcal{M}_{.01}$, $\mathcal{M}_{.10}$, $\mathcal{M}_{.11}$ (Figure 6 in appendix A). As we iterate further, the more future strips and past strips will intersect each other to form more regions which can be labeled as $\mathcal{M}_{S^-.S^+}$ where $S^+ = s_1 s_2 \cdots s_m$ is called the *future itinerary* and $S^- = s_{-n} \cdots s_{-1} s_0$ is called the *past itinerary*. As one may notice, the dynamics of the map is simply acting as a *shift* of the itinerary, where a forward iterate move the entire itinerary to the left through the 'decimal point' and a backward iterate move the entire itinerary to the right. Therefore, we call the set of all bi-infinite itineraries that can be define by $S = S^-.S^+$ the *full shift*.



(a) (b) (c)

FIG. 3: Different iterates of the Hénon map at $(a, b) = (6, -1)$. The two fixed points are marked as red circles. The blue line corresponds to forward iterates, and the green line corresponds to backward iterates. (a) Forward iterate showing $\mathcal{W}^u$, and backward iterate showing $\mathcal{W}^s$. The intersection bounds the state space $\mathcal{M}$, containing the non-wandering set Ω. (b) One more forward iterate. The intersection of $f(\mathcal{M})$ and $\mathcal{M}$ becomes two strips. (c) One more backward iterate. The intersection of $f(\mathcal{M})$ and $f^{-1}(\mathcal{M})$ becomes four regions.

Here we say the Hénon map shows a complete Smale horseshoe because it has a complete binary symbolic dynamics, and as we can see in Fig. 3, every forward fold $f^n(\mathcal{M})$ intersects transversally every backward fold $f^{?m}(\mathcal{M})$. For a given step number m and n, the intersections $\mathcal{M}_{S^-.S^+}$ represents the set of points that do not escape in such forward and backward iterates. Therefore, when m and n goes to infinity, we get the set of points that remain in $\mathcal{M}$ for all time, namely, the non-wandering set of $\mathcal{M}$.

$$\Omega = \left\{ x : x \in \lim_{m,n \to \infty} f^m(\mathcal{M}_.) \bigcap f^{-n}(\mathcal{M}_.) \right\}. \tag{8}$$

However, the non-wandering set for an arbitrary map doesn't necessarily corresponds to the full shift, because some points represented by the itinerary in the full shift may be inadmissible. Thus we say the complete dynamics of the Hénon map is a subshift where all admissible sequences are considered.

The reason why we carry it all the way through to show the Hénon map corresponds to a complete Smale horse is that, a complete Smale horseshoe is *structurally stable*, meaning that all intersections of forward and backward iterates of $\mathcal{M}$ remain transverse for sufficiently small variations of the Hénon map parameters a and b. In a more physical term, the transport properties of the system have a smooth dependence on the parameters. Structural stability is an extremely desirable however rather rare property. A lack of structural stability will results in the creation and destruction of infinitely many periodic orbits for any parameter change, no matter how small it is. For any structurally unstable systems, even as simple as a purely hyperbolic system, any global observable could show a non-smooth dependence on system parameters and behave in a rather unpredictable way. Therefore, the fact that for a specific range of parameters [6] the Hénon map is structural stable is a very important property and what makes any physical interpretation of the Hénon map practically meaningful.

D. Symmetries and Hamiltonian flow

The symmetry of the Hénon is rather simple. For a parameter $b \neq 0$, the Hénon is time reversible with the back ward iteration given by (7), thus giving a b to $1/b$, a to a/b^2 symmetry in the parameter plane, and an x to $-x/b$ symmetry in the coordinate plane.

Some interesting properties for the Hénon map with parameter $b = -1$ can be found. When $b = -1$, the Hénon map (2) becomes

$$a x_n^2 = 1 - x_{n+1} - x_{n-1} \,. \tag{9}$$

The map has an x to x symmetry, the backward and the forward iteration are the same, and according to (5), the area contraction rate is 1. So the map is orientation and area preserving, and the non-wandering set is symmetric about $x_{n+1} = x_n$. Such a simple map corresponds to a Poincaré return map for a 2-dimensional Hamiltonian flow. As we know, for a Hamiltonian flow, the Jacobian matrix $\mathbf{J}$ is a symplectic transformation, $\det \mathbf{J} = 1$ for all the time and the flow is a canonical transformation. This analogue gives us a hint about how we can develop physical interpretations of the Hénon map. In sect. III, we will therefore start with this orientation and area preserving case and discuss a physical interpretation of the Hénon map in detail.

III. PHYSICAL INTERPRETATIONS OF THE HÉNON MAP

As discussed in sect. II D, the area preserving Hénon map is a good starting point to get insightful physical views of the Hénon map. Actually, different interpretations have been proposed based on the connection between the area-preserving Hénon map and the return map of the Hamiltonian flow[2, 3, 7]. In this section, we also start our discussion with the area preserving case, and we choose to review in detail the interpretation demonstrated by Heagy in ref. [3], because in Heagys paper a very comprehensive formulation is developed and more physical meaning has been given to the interpretation instead of just using it as a computational tool to assist mathematical calculations.

A. Area-preserving case

Kicked oscillator systems have been studied as physical model of different maps. For example, a periodically kicked pendulum has been shown to have a return map equivalent to the standard map[8]. In ref. [3], Heagy claimed that the area preserving Hénon map can also be associated with a kicked driven harmonic oscillator with a cubic nonlinear coupling to the kicking term. This association can be derived from the Hamiltonian of the kicked oscillator given by

$$H = \frac{1}{2}p^2 + \frac{1}{2}x^2 + \frac{1}{2}x^3 \sum_{n=-\infty}^{\infty} \delta(t - nT) \,. \tag{10}$$

The Hamilton's equations for the Hamiltonian(10) are

$$\begin{aligned} \frac{\mathrm{d}\,x}{\mathrm{d}\,t} &= p \,, \\ \frac{\mathrm{d}\,p}{\mathrm{d}\,t} &= -x - x^2 \sum_{n=-\infty}^{\infty} \delta(t - nT) \,. \end{aligned} \tag{11}$$

Consider one period of the oscillation T, for a small time interval $(nT, nT + \varepsilon)$ during which the kick takes place, from the Hamiltons equations(11) we have

$$\begin{aligned} \mathrm{d}\,x_n &= x(nT+\varepsilon) - x(nT) = \varepsilon p(nT) \,, \\ \mathrm{d}\,p_n &= p(nT+\varepsilon) - p(nT) = -\varepsilon x(nT) - x(nT)^2 \,. \end{aligned} \tag{12}$$

where the prime denote the variables after the kick. Taking the limit of $\varepsilon \to 0$ we have

$$\begin{aligned} x(nT+\varepsilon) &= x(nT) \,, \\ p(nT+\varepsilon) &= p(nT) - x(nT)^2 \,. \end{aligned} \tag{13}$$

After the kick, for time $nT + \varepsilon \to (n+1)T$, the system goes freely and integrate (10) we get

$$\begin{aligned} x(t) &= C_1 \cos(t - nT) + C_2 \sin(t - nT) \,, \\ p(t) &= -C_1 \sin(t - nT) + C_2 \cos(t - nT) \,. \end{aligned} \tag{14}$$

Substituting (13) into (14) we get $C_1 = x(nT)$ and $C_2 = p(nT) - x(nT)^2$. Substituting C_1, C_2 and using a discrete representation given by $x_n = x(nT), p_n = p(nT)$, equations refeqe:nkick become

$$\begin{aligned} x_{n+1} &= x_n \cos(t-nT) + (p_n - x_n^2)\sin(t-nT)\,,\\ p_{n+1} &= -x_n \sin(t-nT) + (p_n - x_n^2)\cos(t-nT)\,. \end{aligned} \tag{15}$$

This map is already the same as the area-preserving map given by Hénon in ref. [4]. However, if we want to get the standard form of the Hénon map given by (1) or (2), a few more steps need to be taken. First, we write (15) in the two-step recurrent form

$$x_{n+1} + x_{n-1} = 2x_n \cos T - x_n^2 \sin T\,. \tag{16}$$

Then we take the linear transformation $x = \sigma X + \cot T$, where $\sigma = (\cos^2 T - 2\cos T)/\sin T$, and set $a = \cos^2 T - 2\cos T$. Equation (16) becomes

$$X_{n+1} + X_{n-1} = 1 - aX_n^2\,. \tag{17}$$

This is identical to the area-preserving case ($b = -1$) of equation (2). So we see that the Hénon map (1) with $b = -1$ is actually equivalent to the harmonic oscillator system.

B. Dissipative case

Good and straightforward as the derivation for the area-preserving case is, it is also quite limited. Exam the derivation carefully we find the parameter $a = \cos^2 T - 2\cos T$ of the Hénon map is limited to the interval $(-1, 3)$. Therefore, many special chaotic properties of the Hénon map is not accessible, including the whole sequence of period doubling bifurcations that occurs for $a \geq 3$. To better develop the physical interpretation of the complete Hénon map, the dissipative case is also introduced.

Adding a damping term into the oscillator system with a constant damping factor γ, we get the dissipative oscillator system governed by

$$\begin{aligned} \frac{\mathrm{d}\,x}{\mathrm{d}\,t} &= p\,,\\ \frac{\mathrm{d}\,p}{\mathrm{d}\,t} &= -x - \gamma p - x^2 \sum_{n=-\infty}^{\infty} \delta(t-nT)\,. \end{aligned} \tag{18}$$

Similar to the area-preserving case, the effect of the kick in each period is also given by (13). However, the evolution between two kicks is a function of γ. For the time interval $(nT+\varepsilon, (n+1)T)$, the system becomes

$$\begin{aligned} \frac{\mathrm{d}\,x}{\mathrm{d}\,t} &= p\,,\\ \frac{\mathrm{d}\,p}{\mathrm{d}\,t} &= -x - \gamma p\,. \end{aligned} \tag{19}$$

Solving (19) we have

$$\begin{aligned} x_{n+1} =& \mathrm{e}^{-\gamma T/2}(x_n \cos(\omega T)\\ &+ \frac{1}{\omega}(p_n - x_n^2 + \frac{1}{2}\gamma x_n)\sin(\omega T))\,,\\ p_{n+1} =& \mathrm{e}^{-\gamma T/2}[-\omega x_n \sin(\omega T)\\ &+ (p_n - x_n^2 + \frac{1}{2}\gamma x_n)\cos(\omega T) - \frac{1}{2}\gamma x_{n+1}\,. \end{aligned} \tag{20}$$

where $\omega = \sqrt{1 - \frac{1}{4}\gamma^2}$. Again, take a transformation of coordinate

$$\begin{aligned} x =& \sigma_d X + \omega \cot \omega T\,,\\ p =& \frac{\omega \sigma_d}{\sin \omega T}\mathrm{e}^{\gamma T/2}\\ & [1 + \frac{\omega \cot \omega T}{\sigma_d}(1 - \frac{\gamma}{2\omega}\mathrm{e}^{-\gamma T/2}\sin \omega T)\\ & + \mathrm{e}^{-\gamma T/2}(\cos \omega T - \frac{\gamma}{2\omega}\sin \omega T)X + Y] \end{aligned} \tag{21}$$

where

$$\sigma_d = \omega \cot \omega T[\mathrm{e}^{-\gamma T/2}\cos \omega T - (1 + \mathrm{e}^{-\gamma T})]\,. \tag{22}$$

This transformation leads us back to the Hénon map given by (1), with the parameters given by

$$\begin{aligned} a &= \cos \omega T[\mathrm{e}^{-\gamma T}\cos \omega T - \mathrm{e}^{-\gamma T/2}(1 + \mathrm{e}^{-\gamma T})]\,,\\ b &= -\mathrm{e}^{-\gamma T}\,. \end{aligned} \tag{23}$$

Expressing a in terms of b we have

$$a = -\cot \omega T[b \cos \omega T + \sqrt{-b}(1-b)]\,. \tag{24}$$

Therefore, by consider the area-contracting case($b < 1$), parameter a is no more limited to the interval $(-1, 3)$. Yet, this doesn't make all the dynamic properties accessible to the oscillator system. As we have mentioned in sect. II A, for a range of parameters (a, b), the Hénon map have two fixed points. One of the fixed point is always unstable and the other one is at the origin and can be either stable or unstable depending on the choice of parameters. More specifically, the origin is stable for $a < \frac{3}{4}(1-b)^2$. For the dissipative harmonic oscillator case, from (24) we know $a \leq -b + \sqrt{-b}(1-b) \leq \frac{3}{4}(1-b)^2$, so the origin is always stable. However, it is the case where the origin is unstable that shows more interesting properties like period doubling and strange attractor[3].

This stability can be broken by modifying the kick coupling function. Setting the coupling function to be $f(x) = Ax + \frac{1}{3}x^3$ we have

$$\mathrm{d}\,p_n = p(nT+\varepsilon) - p(nT) = -A - x(nT)^2\,. \tag{25}$$

The corresponding return map is given by

$$\begin{aligned} x_{n+1} =& \mathrm{e}^{-\gamma T/2}(x_n \cos(\omega T)\\ &+ \frac{1}{\omega}(p_n - (A + x_n^2) + \frac{1}{2}\gamma x_n)\sin(\omega T))\,,\\ p_{n+1} =& \mathrm{e}^{-\gamma T/2}[-\omega x_n \sin(\omega T)\\ &+ (p_n - (A + x_n^2) + \frac{1}{2}\gamma x_n)\cos(\omega T) - \frac{1}{2}\gamma x_{n+1}\,. \end{aligned} \tag{26}$$

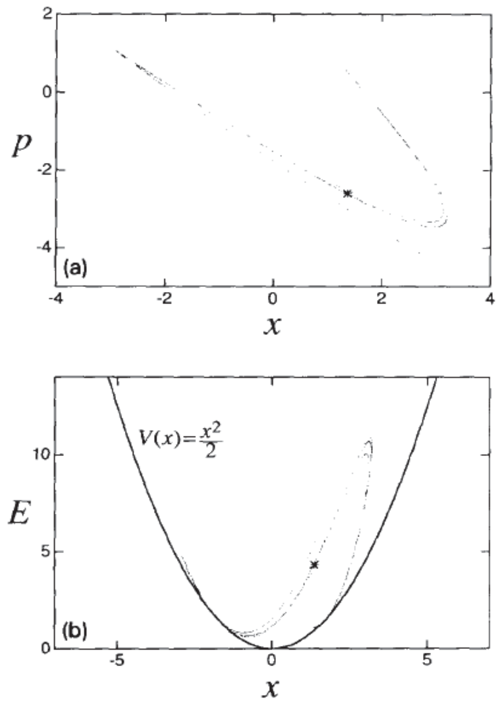


FIG. 4: Figure given in ref. [3]. (a) " strange attractor" of the map with parameters $\gamma = 0.05$, $T = 2.40794509$, and $A = -8.73424$. (b) Energy versus x for the same attractor.

where $\omega = \sqrt{1 - \frac{1}{4}\gamma^2}$. Converting the return map to the standard Hénon map through coordinate transformation we get

$$b = -\mathrm{e}^{-\gamma T}\,,$$
$$a = -b(\cos^2 \omega T - A\frac{\sin^2 \omega T}{\omega^2}) - \sqrt{-b}(1-b)\cos \omega T\,. \tag{27}$$

Since A is unlimited, a is unlimited too. Therefore parameters that make the origin unstable can be reached and properties like the existence of strange attractor can be studied.

In ref. [3], the dissipative harmonic oscillator system with parameters $\gamma = 0.05$, $T = 2.40794509$, and $A = -8.73424$ corresponding to $a = 2.1$ and $b = -0.3$ is studied, and a strange attractor is claimed to be found. Figure 4 shows the "strange attractor" together with the variation of the harmonic potential energy of the same oscillator system with respect to x generated by Heagy in ref. [3].

C. Smooth force driven oscillator

As discussed above, the kick harmonic oscillator shows a good connection with the Hénon map. However, it is attractive to explore whether such properties can be transferred from the impulsively driven oscillator to an oscillator driven by a smooth force, which can relate the Hénon map to much more potential physical applications. In ref. [3], such systems is also briefly discussed.

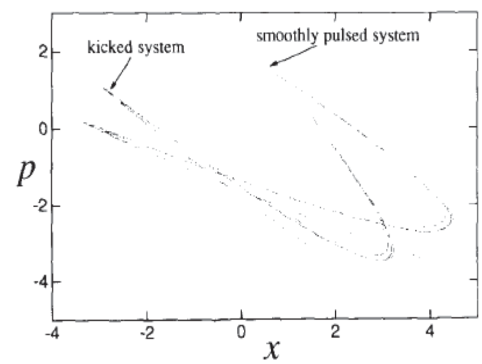


FIG. 5: Figure given in ref. [3]. Strange attractor of kicked system plotted with strange attractor of smoothly pulsed system.

In order to study the smooth force driven harmonic oscillator, the kick coupling function $\delta(t - nT)$ in (10) is replaced by a smooth pulse of finite width given by

$$\begin{aligned} \Delta(t,\varepsilon) =& N(\varepsilon)\exp(\frac{-1}{\varepsilon^2 - t^2})\,, \qquad |t| < \varepsilon\,, \\ & 0\,, \qquad\qquad\qquad\qquad |t| \geq \varepsilon\,. \end{aligned} \tag{28}$$

where the normalization constant is given by

$$N(\varepsilon) = \frac{1}{\varepsilon\sqrt{\pi}U(\frac{1}{2}, 0, 1/\varepsilon^2)}\exp(\frac{1}{\varepsilon^2})\,. \tag{29}$$

Numerical study with parameters $\gamma = 0.05$, $T = 2.40794509$, and $A = -8.73424$ is conducted for the smooth force driven oscillator in ref. [3], and the result is shown in Fig. 5.

The result shows that the smooth force driven system gives a similar attractor structure, and that the impulsive force is not critical for the connection between a harmonic oscillator system and a 'Hénon-like' map. This shows a great potential of such systems to become useful for experimental studies of chaotic dynamics.

IV. SUMMARY AND DISCUSSION

In this paper, we have reviewed the some dynamic properties of the Hénon map. Due to its simply form and interesting chaotic behaviors, the Hénon map has always been attractive as a mathematical model to study deterministic chaos. Besides that, we have also showed that it is possible to find interesting physical interpretation of the Hénon map. The benefits of such an interpretation is manifold. First, the connection between the Hénon map and the harmonic system makes it possible to use the chaotic properties of the Hénon map as a theoretical guideline to explore special properties of such oscillator systems. Second, the physical interpretations of the Hénon map have brought us the possibility to study the dynamic behavior of the Hénon map through experimental studies of specific harmonic systems. Last and most important, the interesting results shown here have provides us a path through which efforts can be put to find connections between mathematical tools and physical systems, thus accelerate the exploration of both.

Indeed, there are a lot more topics related to this study can be and worth being investigated. For the Hénon map, more discussion about ways to find periodic orbits and possible ways to relate them to the physical behavior of harmonic oscillator systems would be interesting. And regarding the map of the smooth force driven oscillator, more detailed study showing its dynamics like the cycles and bifurcations would be instructive. Furthermore, if we broaden our extent of study to other simple nonlinear maps and other interesting physical systems, we may find more hidden connections between existing mathematical models and physical systems, and interesting progresses may be made. For example, nonlinearity has always been a concern for resonating MEMS devices development. Although nonlinear behaviors have been shown to be beneficial in some applications, such as increasing the bandwidth of inertial sensors[9] and reducing temperature dependence of frequency of MEMS resonators[10], due to the lack of systematical studies of the chaotic behaviors of micro-mechanical systems, nonlinearity is avoided intentionally in most MEMS designs. If an instructive connections between such systems and any specific nonlinear models can be found, and solid theoretical base of the chaotic properties can be developed, then fantastic applications will become possible. To summarize, the Hénon map is not only the simplest map to study chaotic dynamics, it also shows potentials to solve problems in the physical world. And the exploration of physical interpretations of simply nonlinear maps like the Hénon map is more than a trivial thing. A good physical interpretation of a mathematical system can be beneficial for both the study of mathematics and physics.

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[1] M. Hénon, "A two-dimensional mapping with a strange attractor," *Comm. Math. Phys.* **50** (1976) 69.

[2] O. Biham and W. Wenzel, "Characterization of unstable periodic orbits in chaotic attractors and repellers" *Phys. Rev. Lett.* **63** (1989) 819.

[3] J.F. Heagy, "A physical interpretation of the Hénon map," *Phys. D* **57** (1992) 436-446.

[4] M. Hénon, "Numerical study of quatratic area-preserving mappings," *Quart. Appl. Math.* **27** (1969) 291.

[5] W. F. H. Al-Shameri, "Dynamical properties of the Hénon mapping," *Int. Journal of Math. Analysis, Vol. 6* **49** (2012) 2419-2430.

[6] D. Sterling, H. R. Dullin, J. D. Meiss, "Homoclinic bifurcations for the Hénon map," *Phys. D* **134** (1999) 153-184.

[7] R.H.G. Helleman, "Self-generated chaotic behaviour in nonlinear mechanics," in *Fundamental problems in statistical mechanics, Vol. 5,* ed. E.G.D. Cohen, (North-Holland, Amsterdam, 1980).

[8] B. V. Chirikov, "A univeral instability of many dimensional oscillator systems," *Phys. Rep. Vol. 52* **5** (1979) 263-379.

[9] R. Lifshitz and M. C. Cross, Chapter "Nonlinear dynamics of nanomechanical and micromechanical resonators," in *Reviews of Nonlinear Dynamics and Complexity,* ed. H. G. Schuster, (Wiley-VCH Verlag GmbH & Co. KGaA, Weinheim, Germany 2009).

[10] R. Tabrizian, G. Casinovi, and F. Ayazi, "Temperature stable silicon oxide (SilOx) micromechanical resonators," *IEEE Trans. on Elec. Dev. Vol. 60,* **8** (2013) 2656-2663.

[11] P. Cvitanović, R. Artuso, R. Mainieri, G. Tanner and G. Vattay, *Chaos: Classical and Quantum*, `ChaosBook.org` (Niels Bohr Institute, Copenhagen 2012).

APPENDIX A: ADDITIONAL FIGURES

(a) (b)

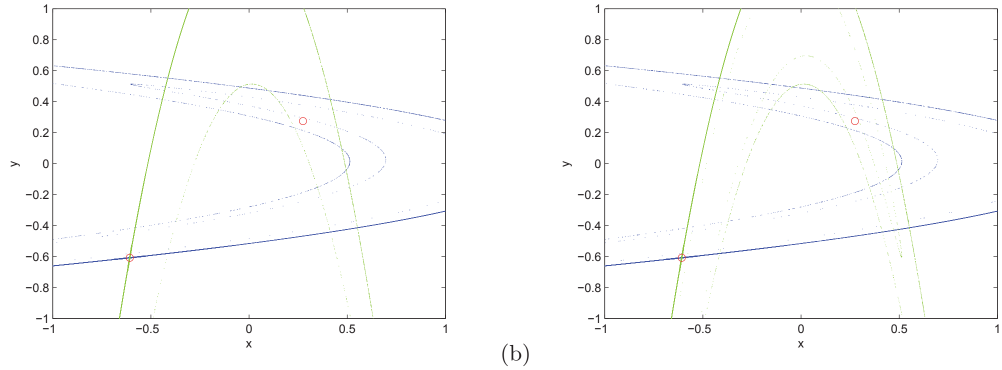

FIG. 6: Different iterates of the Hénon map at $(a, b) = (6, -1)$. The two fixed points are marked as red circles. The blue line corresponds to forward iterates, and the green line corresponds to backward iterates. (a) Two more forward iterate of $\mathcal{M}$. (b) Two more backward iterate of $\mathcal{M}$.

APPENDIX B: MATLAB CODES

Henon.m

```
%The Henon Map
function x1 = Henon( x0,a,b )
x1(1,1)=1-a*x0(1,1)^2+b*x0(1,2);
x1(1,2)=x0(1,1);
end
```

Henon_iver.m

```
%Backward Henon Map
function x0 = Henon_inver( x1,a,b )
x0(1,1)=x1(1,2);
x0(1,2)=-1/b*(1-a*x1(1,2)^2-x1(1,1));
end
```

trjactory.m

```
%% Henon Attractor
clc;
clear all;
a=1.4; b=0.3;
n=10000;
init=[0.1; 0.1];
x=zeros(n,2);
x(1,:)=init;
c=0;
for i=1:n-1
     x(i+1,:)=Henon(x(i,:),a,b);
     c=c+1;
    plot(x(i,1),x(i+1,1));
    hold on;
%      if c<1000
%          plot(x(i,1),x(i+1,1),'b*');
%          hold on;
%      else
%          hold off;
%          c=0;
```

```
%     end
end
xlabel('x_n');
ylabel('x_n_+_1');
```

horseshoe.m

```
%% Henon Map Horseshoe
clc;
clear all;
a=6; b=-1;
x_f1=(-(1-b)-sqrt((1-b)^2+4*a))/(2*a);
x_f2=(-(1-b)+sqrt((1-b)^2+4*a))/(2*a);
n=10000;
x0=x_f1-0.01+0.02*rand(n,2);
init_step=4;
forward=init_step+1;
backward=init_step+1;
for i=1:forward
    for j=1:n
    x1(j,:)=Henon(x0(j,:),a,b);
     if x1(j,1)<1&&x1(j,1)>-1&&x1(j,2)<1&&x1(j,2)>-1
        plot(x1(j,1),x1(j,2));
        hold on;
     end
    end
    x0=x1;
end
x0=x_f1-0.01+0.02*rand(n,2);
for i=1:backward
    for j=1:n
    x_1(j,:)=Henon_inver(x0(j,:),a,b);
     if x_1(j,1)<1&&x_1(j,1)>-1&&x_1(j,2)<1&&x_1(j,2)>-1
        plot(x_1(j,1),x_1(j,2),'g');
        hold on;
     end
    end
    x0=x_1;
end
plot(x_f1,x_f1,'or',x_f2,x_f2,'or');
xlabel('x');
ylabel('y');
```